Lecture 15

Zeros of H(z) and the Frequency Domain

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READING ASSIGNMENTS

• This Lecture:
  – Chapter 7, Section 7-6 to end

• Other Reading:
  – Recitation & Lab: Chapter 7
    • ZEROS (and POLES)
  – Next Lecture: Chapter 8

LECTURE OBJECTIVES

• ZEROS and POLES
• Relate \( H(z) \) to FREQUENCY RESPONSE
  \[ H(e^{j\theta}) = H(z) \bigg|_{z=e^{j\theta}} \]

• THREE DOMAINS:
  – Show Relationship for FIR:
    \[ h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\theta}) \]

DESIGN PROBLEM

• Example:
  – Design a Lowpass FIR filter (Find \( b_k \))
  – Reject completely \( 0.7\pi, 0.8\pi, \) and \( 0.9\pi \)
    • This is NULLING
  – Estimate the filter length needed to accomplish this task. How many \( b_k \)?

• Z POLYNOMIALS provide the TOOLS
**Z-Transform**

- **Definition**: Polynomial Representation of LTI System:
  
  \[ H(z) = \sum_{n} h[n] z^{-n} \]

- **Example**:
  \[ \{ h[n] \} = \{ 2,0,-3,0,2 \} \]
  \[
  H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4} \\
  = 2 - 3z^{-2} + 2z^{-4} \\
  = 2 - 3(z^{-1})^2 + 2(z^{-1})^4
  \]

**Convolution Property**

- Convolution in the \( n \)-domain
  - **Same as**
  - Multiplication in the \( z \)-domain

\[ y[n] = h[n] * x[n] \iff Y(z) = H(z)X(z) \]

\[ y[n] = x[n] * h[n] \iff y[n] = \sum_{k=0}^{M} h[k] x[n-k] \]

**Convolution Example**

\[ x[n] \quad H(z) \quad y[n] \]

\[
\begin{align*}
  x[n] &= \delta[n-1] + 2\delta[n-2] \\
  h[n] &= \delta[n] - \delta[n-1] \\
  y[n] &= x[n] * h[n] \\
  H(z) &= 1 - z^{-1} \\
  Y(z) &= (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3} \\
  y[n] &= \delta[n-1] + \delta[n-2] - 2\delta[n-3]
\end{align*}
\]

**Three Domains**

- **Z-Transform Domain**
  - **Polynomials: H(z)**

- **Time Domain**
  - **\( \{ b_k \} \)**

- **Frequency Domain**
  - **\( H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} \)**

**Frequency Response?**

- Same Form:
  \[ \hat{\omega} - \text{Domain} \]
  \[ H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega} k} \]
  \[ \hat{z} = e^{j\hat{\omega}} \]

- **\( z \)-Domain**
  \[ H(z) = \sum_{k=0}^{M} b_k z^{-k} \]

**Another Analysis Tool**

- **\( z \)-Transform Polynomials are Easy!**
  - **Roots, Factors, etc.**

- **Zeros and Poles**: Where is \( H(z) = 0 \) ?

- The \( z \)-domain is **Complex**
  - \( H(z) \) is a **Complex-Valued** function of a **Complex Variable** \( z \)
ZEROS of $H(z)$

- Find $z$, where $H(z) = 0$
  
  \[
  H(z) = 1 - \frac{1}{2} z^{-1}
  \]

  \[1 - \frac{1}{2} z^{-1} = 0 \implies z - \frac{1}{2} = 0\]

  Zero at $z = \frac{1}{2}$

POLES of $H(z)$

- Find $z$, where $H(z) = 0$
  
  - Interesting when $z$ is ON the unit circle.

  \[
  H(z) = 1 - 2 z^{-1} + 2 z^{-2} - z^{-3}
  \]
  \[
  H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})
  \]

  Roots: $z = 1, \frac{1}{2} \pm j \frac{\sqrt{3}}{2}, e^{\pm j\pi/3}$

PLOT ZEROS in z-DOMAIN

FREQ. RESPONSE from ZEROS

- Relate $H(z)$ to FREQUENCY RESPONSE
- EVALUATE $H(z)$ on the UNIT CIRCLE
  - ANGLE is the same as FREQUENCY

\[z = e^{j\hat{\omega}} \text{ (as } \hat{\omega} \text{ varies)}\]

defines a CIRCLE, radius = 1

\[
H(e^{j\hat{\omega}}) = H(z)\bigg|_{z = e^{j\hat{\omega}}}
\]
nulling property of $H(z)$

- When $H(z) = 0$ on the unit circle.
  - Find inputs $x[n]$ that give zero output

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\theta}) = 1 - 2e^{-j\theta} + 2e^{-2j\theta} - e^{-3j\theta}$$

- Evaluate $H(z)$ at the input “frequency” $\omega$

$$H(e^{j\frac{\pi}{3}}) = 1 - 2e^{-j\frac{\pi}{3}} + 2e^{-2j\frac{\pi}{3}} - e^{-3j\frac{\pi}{3}} = 0$$
DESIGN PROBLEM

• Example:
  – Design a Lowpass FIR filter (Find \( b_k \))
  – Reject completely 0.7\( \pi \), 0.8\( \pi \), and 0.9\( \pi \)
  – Estimate the filter length needed to accomplish this task. How many \( b_k \)?

• Z POLYNOMIALS provide the TOOLS

CASCADE EQUIVALENT

• Multiply the System Functions

\[
H(z) = H_1(z)H_2(z)
\]

CASCADE EXAMPLE

\[
\begin{align*}
H_1(z) &= 1 - z^{-1} \\
H_2(z) &= 1 + z^{-1} \\
y[n] &= x[n] - x[n - 1] - w[n] + w[n - 1]
\end{align*}
\]

L-pt RUNNING SUM \( H(z) \)

\[
H(z) = \sum_{k=0}^{L-1} \frac{z^{-k}}{1 - z^{-1}} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^{L-1} - 1}{Lz^{L-1}(z - 1)}
\]

\( z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{2\pi jL/k} \)

for \( k = 1, 2, \ldots, L - 1 \)

ZEROS on UNIT CIRCLE

[2-1] in denominator cancels k=0 term

11-pt RUNNING SUM \( H(z) \)

\[
H(z) = \sum_{k=0}^{10} z^{-k}
\]

\[
H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1}) \cdots (1 - e^{j20\pi/11}z^{-1})
\]

NO zero at \( z=1 \)
**FILTER DESIGN: CHANGE L**

Passband is Narrower for L bigger

![Filter Design Diagram](image)

**NULLING FILTER**

- PLACE ZEROS to make \( y[n] = 0 \)

\[
H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}
\]

the output resulting from each of the following three signals will be zero:

- \( x_1[n] = (z_1)^n = 1 \)  \( y_1[n] = 0 \)
- \( x_2[n] = (z_2)^n = e^{j\pi/3} \)  \( y_2[n] = 0 \)
- \( x_3[n] = (z_3)^n = e^{-j\pi/3} \)  \( y_3[n] = 0 \)

**3 DOMAINS MOVIE: FIR**

![3 Domains Movie Diagram](image)

**POP QUIZ: MAG & PHASE**

- Given: \( H(e^{j\theta}) = e^{-j\theta} \cos(\theta) \)
- Derive Magnitude and Phase

\[
|H(e^{j\theta})| = |e^{-j\theta}|, |\cos(\theta)| = \cos(\theta)
\]

\[
\angle H(e^{j\theta}) = \begin{cases} \hat{\theta} \quad \cos(\hat{\theta}) \geq 0 \\ \hat{\theta} + \pi \quad \cos(\hat{\theta}) < 0 \end{cases}
\]

**POP QUIZ : Answer #2**

- Find \( y[n] \) when

\[
x[n] = \cos(0.25\pi n)
\]

\[
y[n] = |H|\cos(0.25\pi n + \angle H) = 0.707\cos(0.25\pi n - \frac{\pi}{4})
\]

\[
H(e^{j\theta}) = e^{-j\theta} \cos(\theta) \quad \text{at } \hat{\theta} = \frac{\pi}{4}
\]

\[
H(e^{j\pi/4}) = e^{-j\pi/4} \cos\left(\frac{\pi}{4}\right) = 0.707e^{-j\pi/4}
\]

**Ans: FREQ RESPONSE**

![Frequency Response Diagram](image)
CHANGE in NOTATION

- Relate $H(z)$ to FREQUENCY RESPONSE

$$H(\hat{\omega}) = H(z)\bigg|_{z = e^{j\hat{\omega}}}$$

NEW NOTATION for FREQUENCY RESPONSE

$$H(\hat{\omega}) \leftrightarrow H(e^{j\hat{\omega}})$$